

The Fourier transform of the Hadamard transform: Multifractals, Sequences and Quantum Chaos

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Abstract— We introduce a class of functions that limit to multifractal measures and which arise when one takes the Fourier transform of the Hadamard transform. This introduces generalizations of the Fourier transform of the well-studied and ubiquitous Thue-Morse sequence, and introduces also generalizations to other intriguing sequences. We show their relevance to quantum chaos, by displaying quantum eigenfunctions of the quantum bakers map that are approximated well by such measures, thereby extending our recent work where we pointed to the existence of “Thue-Morse” states.

Keywords—Fourier transform. Hadamard transform. Multifractals. Sequences. Quantum chaos. Eigenfunctions.

I. INTRODUCTION

THE Fourier and Hadamard transforms are standard tools, widely used in science and signal processing [1]. The relative importance of the two transforms may be judged to be a factor of thirty in favour of the Fourier transform if one were to go by a “google” search which returned over five million webpages for this transform. Both these transforms can be implemented with fast algorithms that reduce their implementation on N data points from N^2 to $N \log(N)$ operations. The fast Fourier transform (FFT) and the fast Hadamard transform essentially rely on the factoring of the transform into operators acting on product vector spaces. The Hadamard transform though is a real transform which only adds or subtracts the data and is therefore widely used in digital signal processing. The Fourier transform conjugate spaces are familiar ones (“time-frequency”, “position-momentum”) etc., while the corresponding Hadamard transforms are not so well understood. Nevertheless the Hadamard transform has also received great attention in the recent past due to its uses in quantum computing, with the Hadamard gate being a central construct [2].

We define the Fourier transform (FT) on N sites as

$$(G_N)_{m,n} = \frac{1}{\sqrt{N}} \exp(-2\pi i(m+\alpha)(n+\alpha)/N) \quad (1)$$

with $0 \leq \alpha \leq 1/2$ being a phase parameterizing the transform, and $0 \leq m, n \leq N-1$. G_N as a matrix is an unitary one. The Hadamard transform that we use maybe written in several ways, firstly as an tensor or Kronecker product form, secondly via a recursion and finally via their matrix elements. In all of this and what follows in this paper we assume N to be a power of 2, i.e., $N = 2^K$, for some integer K . If

$$H_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \quad (2)$$

then

$$H_{2^K} = H_2 \otimes H_2 \otimes \cdots \otimes H_2 = \otimes^K H_2. \quad (3)$$

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Equivalently

$$H_{2^{i+1}} = \frac{1}{\sqrt{2}} \begin{pmatrix} H_{2^i} & H_{2^i} \\ H_{2^i} & -H_{2^i} \end{pmatrix}, \quad (4)$$

and $H_{2^0} = 1$. Also in terms of matrix elements

$$(H_N)_{m,n} = \frac{1}{\sqrt{N}} (-1)^{a \cdot b} \quad (5)$$

where $a \cdot b = \sum_{i=1}^K a_i b_i$ and $m = \sum_{i=1}^K a_i 2^{i-1}$, $n = \sum_{i=1}^K b_i 2^{i-1}$, that is a and b are vectors whose entries are the binary expansions of the matrix positions (m, n) . Note that H_N is such that $H_N^2 = I$, while $G_N^4 = I$, where I is the identity, therefore the spectrum of both these transforms are highly degenerate (± 1 for H_N , $\pm 1, \pm i$ for G_N).

Also notice that we can enumerate the columns of H_N (or rows, as it is a symmetric matrix) as outer products of

$$v_0 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad v_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \quad (6)$$

If $n = a_{K-1}a_{K-2}\cdots a_0$ is its binary representation ($0 \leq n \leq 2^K - 1$), the n^{th} column V_n is the outer product

$$V_n = v_{a_{K-1}} \otimes v_{a_{K-2}} \otimes \cdots \otimes v_0. \quad (7)$$

While the first column V_0 is simply an uniform string of 1s, the last one V_{N-1} is the K -th generation of the Thue-Morse sequence.

The Thue-Morse sequence is an example of an “automatic sequence” [3] of two alphabets say A and B . Given say A , the rule is to replace it by AB and given B replace it by BA . Thus starting with the seed A we get $\{A \rightarrow AB \rightarrow ABBA \rightarrow ABBABAAB \rightarrow \dots\}$. The K -th generation consists of a string or word of length 2^K which is cube free, that is no block (any finite string consisting of the two alphabets) repeats thrice consecutively. This sequence occurs in numerous contexts [3], combinatorics on words, number theory, group theory, dynamical systems, to name a few, and is considered to be marginal between a quasiperiodic sequence and a chaotic one. The deterministic disorder of this sequence is relevant to models of quasicrystals [4], mesoscopic disordered system [5], and as we established recently to quantum chaos [6]. The column V_{N-1} is got from the K -th generation of the Thue-Morse sequence via the identification $A \equiv 1$ and $B \equiv -1$, apart from the factor $1/\sqrt{N}$.

Our recent work indicated that the Fourier transform of the Thue-Morse sequence, along with the sequence itself was an excellent ansatz for a class of eigenstates in the quantum baker map, which we called the “Thue-Morse states”. The classical baker map is a paradigmatic and simple model of complete

Hamiltonian chaos. Its quantization is then of considerable interest in the study of quantum chaos, therefore the emergence of this ansatz provides an interesting way to think of the deterministic structural disorder of quantum chaotic eigenfunctions, at least in this model system. We also found that the Fourier transform of some other columns of the Hadamard transform played a crucial role in describing other states. The Fourier transform of the Thue-Morse sequence [7], [9], or some of the other columns of H_N are not simple functions though, they could be multifractals [8]. Of course the Fourier transform of the first column V_0 is just a localized delta peak (which maybe broadened for nonzero α); thus we expect that the Fourier transform of the Hadamard matrix will result in a mixture of functions or measures with a range of complexity. This is our primary motivation for studying the product $G_N H_N$, the Fourier transform of the Hadamard transform.

II. THE MATRIX ELEMENTS OF $G_N H_N$

The matrix elements of $G_N H_N$ are evaluated economically as a product of K trigonometric terms. Using the matrix representations of G_N and H_N we get

$$(G_N H_N)_{kn} = \frac{1}{N} \sum_{l=0}^{N-1} e^{-2\pi i(k+\alpha)(l+\alpha)/N} (-1)^{\sum_{j=0}^{K-1} b_j a_j} \quad (8)$$

where $l = \sum_{j=0}^{K-1} 2^j b_j$, and $n = \sum_{j=0}^{K-1} 2^j a_j$. Thus performing the independent sums over the b_j , and after some simplifications, we get

$$(G_N H_N)_{kn} = e^{-i\pi(k+\alpha)(N-1+2\alpha)/N} e^{-i\pi \sum_{j=0}^{K-1} a_j/2} \times \prod_{j=0}^{K-1} \cos \left[\frac{\pi}{N}(k+\alpha)2^j + \frac{\pi}{2}a_j \right] \quad (9)$$

Thus we are led to the study of the following class of functions which are power spectra's of the columns of the Hadamard matrix: $|(G_N V_n)_k|^2 = |(G_N H_N)_{kn}|^2 \equiv f_n(k)$

$$f_n(k) = \prod_{j=0}^{K-1} \cos^2 \left[\frac{\pi}{N}(k+\alpha)2^j + \frac{\pi}{2}a_j \right] \quad (10)$$

We view these as a function of k for a fixed $n = a_{K-1}a_{K-2}\dots a_0$. They satisfy the normalization, that follows from the unitarity of $G_N H_N$,

$$\sum_{k=0}^{N-1} f_n(k) = 1 \quad (11)$$

and we will in fact treat $f_n(k)$ as a probability measure. We are interested in the limit $N \rightarrow \infty$ (or $K \rightarrow \infty$). If there exist sequences of n that lead to limiting distributions we are especially interested in these. In the following we use the notation that $(s)_m$ is an m -fold repetition of the binary string s . If n is of this form we also denote $f_n(k)$ as $f_{(s)}(k)$. For instance the case when $n = N - 1 = (1)_K$ leads to the power spectrum of the Thue-Morse sequence that is well-known to limit to an

multifractal measure [7]. In this case

$$f_{(1)}(k) = \prod_{j=0}^{K-1} \sin^2 \left[\frac{\pi}{N}(k+\alpha)2^j \right]. \quad (12)$$

This has been particularly studied when $\alpha = 0$ and found to limit to a multifractal with a correlation dimension $D_2 = 0.64$ [7].

III. THE PARTICIPATION RATIO OR THE CORRELATION DIMENSION

To probe the limiting functions, if they exist, for multifractality we test for the scaling relation

$$P_n^{-1} = \sum_{k=0}^{N-1} f_n^2(k) \sim N^{-D_2}. \quad (13)$$

The left hand side of this is also interpreted as the inverse participation ratio, its inverse, P_n , being the effective spread of the power spectrum, an estimate of the number of “frequencies” (k) that participate in it. If $D_2=0$, the frequencies are localized (Bragg peaks of crystallography), if $D_2=1$, it is a situation expected of power spectra's of random sequences. In the intermediate range are multifractals, with structures at many scales.

Consider the class of functions that result as K tends to ∞ along even numbers, and $n = (01)_{K/2}$. Equivalently $n = (N-1)/3$, and the functions are $f_{(01)}(k)$. We also simultaneously consider the closely related functions $f_{(10)}(k)$. Both these tend to multifractal measures as $K \rightarrow \infty$ with $D_2 \approx 0.57$. The principal peaks of $f_{(10)}(k)$ are at $1/5, 4/5$, while those of $f_{(01)}(k)$ are at $2/5, 3/5$. Taken together these peaks constitute a *period-4* orbit of the doubling map $x \mapsto 2x \pmod{1}$. The peaks of the Fourier transform of the Thue-Morse sequence, or of $f_{(1)}(k)$ are at the period-2 orbits of the doubling map, i.e. at $1/3, 2/3$. We see these and a few other such functions in Fig. (1). The scaling of the participation ratio of these measures and the corresponding correlation dimension are shown in Fig. (2), which shows that indeed these measures are multifractals. In the case of strings of the form (001) for instance, K is taken to be multiples of 3 and so on. One interesting observation from this figure is that it appears that the more 1 there are in the string s , the more is the dimension D_2 , so that the power-spectrum of the Thue-Morse sequence may have the maximum possible D_2 value in this class of multifractals. Of course the string $(0)_K$ is not a multifractal at all, and $D_2 = 0$ in this case.

For a given N , the participation ratio P_n of the Fourier transform for the various columns n , has a range of values that indicates the localization in the conjugate basis. We show in Fig. (3) the participation ratios for the case $N = 1024$ and $\alpha = 1/2$. We see here the intricate way in which the columns of the Hadamard matrix are arranged. The largest participation ratio occurs for the last column of the Hadamard matrix which corresponds to the Thue-Morse sequence. We find a similar behaviour for other values of α , as well as other measures of localization such as the entropy. We include this parameter as for the application we have in mind, namely the quantum bakers map, $\alpha = 1/2$ is pertinent.

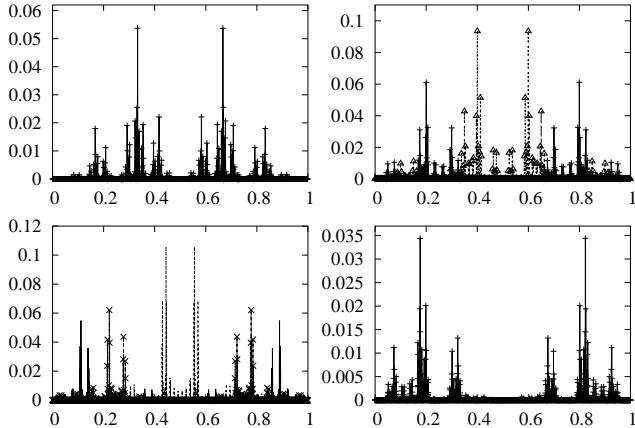


Fig. 1

THE FRACTAL MEASURES $f_{(1)}$, $(f_{(01)}, f_{(10)})$, $f_{(110)}$, $(f_{(001)}, f_{(010)}, f_{(100)})$ ARE PLOTTED CLOCKWISE. $K = 10$ IN THE FIRST CASE AND $K = 12$ IN THE OTHERS, WHILE $\alpha = 0$ UNIFORMLY. WE HAVE GROUPED SOME FUNCTIONS AND SHOWN THEM IN DIFFERENT STYLES IN THE FIGURE.

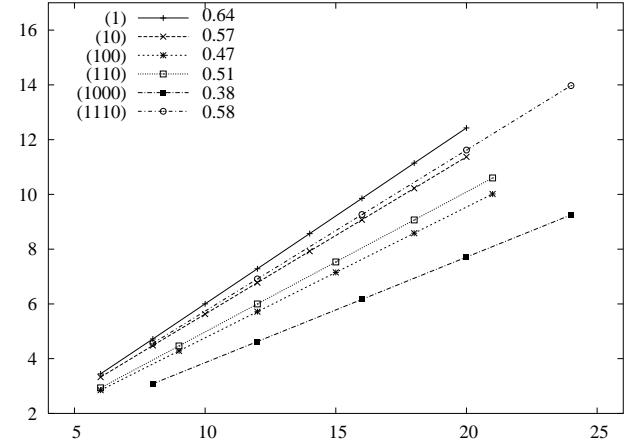


Fig. 2

THE SCALING OF THE PARTICIPATION RATIO P_n WITH N . $\log_2(P_n)$ IS PLOTTED AGAINST $\log_2(N) = K$, AND $n = (s)_m$, WHERE THE STRING s IS GIVEN IN THE KEY TO THE FIGURE, AND m IS A WHOLE NUMBER SUCH THAT K/m IS THE LENGTH OF THE STRING s . THE SLOPES OF THE LINES ARE ALSO GIVEN IN THE KEY.

We now briefly discuss the sequence whose power spectrum is $f_{(01)}(k)$. Recall that the corresponding sequence for $f_{(1)}(k)$ was the Thue-Morse sequence $\{1, -1, -1, 1, -1, 1, 1, -1, \dots\}$. The n -th term of this sequence is $t_n = (-1)^{\sum_{i=0}^{K-1} a_i}$ where again the a_i are bits of the binary expansion of n . Similarly the n -th term of the sequence whose power spectrum is $f_{(01)}(k)$ is

$$t_n = (-1)^{\sum_{i=0,2,\dots} a_i}. \quad (14)$$

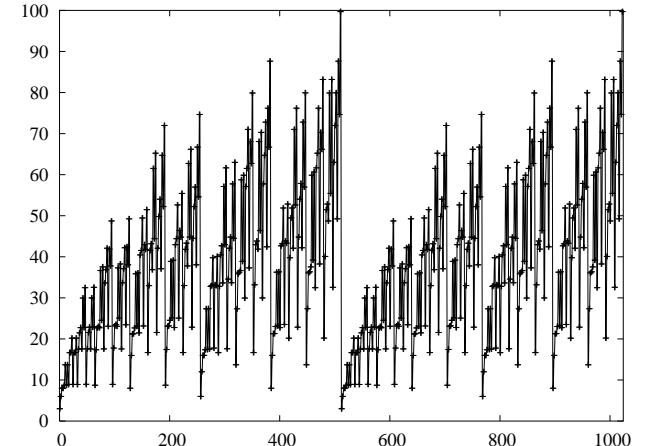
The first few terms of this sequence are $\{1, -1, 1, -1, -1, 1, -1, 1, 1, -1, 1, -1, -1, 1, -1, 1, -1, 1, -1, 1, \dots\}$. To write an concatenation rule we use the fact that this is formed by repeated outer product of $(1, -1, 1, -1)^T$ and get

$$S(k+1) = S(k) \overline{S(k)} S(k) \overline{S(k)} \quad (15)$$

Where $S(k)$ is the k -th generation of the sequence, with $S(0) = 1$, and $\overline{S(k)}$ is the complementary set where 1 and -1 are interchanged. While we found similar rules and sequences elsewhere [10], we did not find this exact one. It also appears that the inflation rules $A \rightarrow ABAB$, $B \rightarrow BABA$ produces this sequence.

IV. CONNECTIONS TO QUANTUM CHAOS

So far we have introduced the measure and discussed some of their mathematical properties. Here we make explicit their relevance to quantum chaos. The classical baker's map [11], T , is the area preserving transformation of the unit square $[0, 1] \times [0, 1]$ onto itself, which takes a phase space point (q, p) to (q', p') where $(q' = 2q, p' = p/2)$ if $0 \leq q < 1/2$ and $(q' = 2q - 1, p' = (p+1)/2)$ if $1/2 \leq q < 1$. The stretching along the horizontal q direction by a factor of two is compensated exactly by a compression in the vertical p direction. The repeated action of T on the square leaves the phase space mixed.

Fig. 3
THE PARTICIPATION RATIO P_n VS n FOR $N = 1024$ AND $\alpha = 1/2$.

this is well known to be a fully chaotic system that in a mathematically precise sense is as random as a coin toss. The area-preserving property makes this map a model of chaotic two-degree of freedom Hamiltonian systems, and the Lyapunov exponent is $\log(2)$ per iteration.

The baker's map was quantized by Balazs and Voros [12], while Saraceno [13] imposed anti-periodic boundary conditions, and this leads to the quantum baker's map, in the position representation, that we use in this Letter:

$$B = G_N^{-1} \begin{pmatrix} G_{N/2} & 0 \\ 0 & G_{N/2} \end{pmatrix}, \quad (16)$$

where $(G_N)_{mn} = \langle p_m | q_n \rangle = \exp[-2\pi i(m + 1/2)(n + 1/2)/N]/\sqrt{N}$. The Hilbert space is finite dimensional, the di-

mensionality N being the scaled inverse Planck constant ($N = 1/h$), where we have used that the phase-space area is unity. The position and momentum states are denoted as $|q_n\rangle$ and $|p_m\rangle$, where $m, n = 0, \dots, N-1$ and the transformation function between these bases is the finite Fourier transform G_N given above.

The choice of anti-periodic boundary conditions fully preserves parity symmetry, here called R , which is such that $R|q_n\rangle = |q_{N-n-1}\rangle$. Time-reversal symmetry is also present and implies in the context of the quantum baker's map that an overall phase can be chosen such that the momentum and position representations are complex conjugates: $G_N\phi = \phi^*$, if ϕ is an eigenstate in the position basis. B is an unitary matrix, whose repeated application is the quantum version of the full left-shift of classical chaos. There is a semiclassical trace formula, which, based on the unstable periodic orbits, approximates eigenvalues [14].

The nature of quantum chaotic eigenfunctions is intriguing as they display a bewildering variety of patterns, that can sometimes be partially attributed to classical periodic orbits. This phenomenon called scarring [15] is apparently in conflict with another observed behaviour namely their similarity to random matrix eigenfunctions [16]. The quantum bakers map affords us an opportunity to study such states in a simple setting, however no known analytical formulae exist. We had proposed a few ansatz for a variety of states based upon the Thue-Morse sequence and its Fourier transform [6]. These could sometimes reproduce states to more than 99%. In this work we point to other measures such as $f_{(01)}$ that also play a role in the spectrum of the quantum bakers map.

In Fig. (4) we see two examples of states of the quantum bakers map in the position basis for $N = 1024$, along with their Hadamard transforms. The Hadamard transforms are particularly simple, the Thue-Morse state, ψ_{tm} , being the first [6]. We show it here for comparison with another state whose Hadamard transform has distinct peaks at around $N/3$ and $2N/3$ implying that it is likely that the measures $f_{(01)}$ and $f_{(10)}$ discussed above are relevant for these states. There are usually more than one such state, we show here a particularly "clean" state, in terms of its Hadamard transform, and for the purpose of this paper call it $\psi_{(01)}$.

Indeed we can construct the ansatz

$$\phi = (\gamma_1 + \gamma_1^* G^{-1}) V_{(N-1)/3} + (\gamma_2 + \gamma_2^* G^{-1}) V_{2(N-1)/3} \quad (17)$$

where N are powers of 4. This includes both the columns at $(N-1)/3$ and $2(N-1)/3$, along with their Fourier transforms. The latter are included due to the presence of time-reversal symmetry. In fact they dominate the state and is the motivation for our study. This ansatz has two complex constants γ_1 and γ_2 which we determine numerically so that its overlap with the actual state $\psi_{(01)}$ is the maximum. For instance in the case $N = 64$, we were able to find γ such that the overlap $|\langle \phi | \psi_{(01)} \rangle|^2 \approx 0.75$. A comparison of the spectral measures for the Thue-Morse sequence, as well as $f_{(01)}$ and $f_{(10)}$ in Fig. (1) with the actual eigenfunctions in Fig. (4) show the similarity between them. Notice that the actual wavefunction is more closely related to the Fourier transforms of a linear combination of the columns of the Hadamard matrix. The peaks of

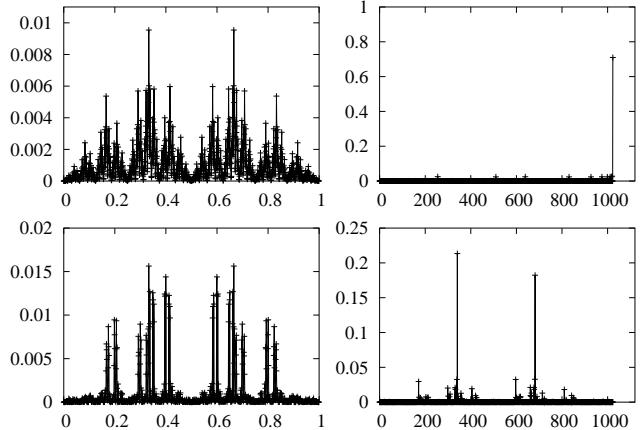


Fig. 4

THE THUE-MORSE STATE ψ_{tm} (TOP LEFT) AND A STATE CALLED IN THE TEXT $\psi_{(01)}$ (BOTTOM LEFT), ALONG WITH THEIR CORRESPONDING HADAMARD TRANSFORMS (RIGHT). IN THIS CASE $N = 1024$ AND THE PEAKS OF THE HADAMARD TRANSFORM OF $\psi_{(01)}$ OCCUR AT $(N-1)/3$ AND $2(N-1)/3$.

this wavefunction can be identified with the period-4 orbit starting from $(q = 1/5, p = 4/5)$, and has been noted earlier the q part of this corresponds to the peaks in the measures $f_{(01)}$ and $f_{(10)}$.

We have concentrated on convergent measures as $n = (s)_m$ tends to infinity and s is a bit string repeated m times. We can have other sequences of n that lead to convergent measures, in particular those whose binary expansions ends in a string of 1 or 0. In the former case we have already used such measures to describe a family of states of the bakers map that are scarring by period-2, period-4 and associated homoclinic orbits [6]. As spectral measures they are different depending on the finite string that precedes the infinite string of 1; however their correlation dimension D_2 seems to be the same as that of the spectrum of the Thue-Morse sequence, namely 0.64. As we noted earlier this seems to be consistent with our view that more the 1, more the dimension D_2 . If the string a_i is itself not periodic, say when n is tending to infinity such that $n/2^K$ tends to an irrational number, there does not seem to be any convergent measures.

In summary, we have introduced a class of simple functions that limit to multifractals, have interesting connections to sequences and to a simple model of quantum chaos. We have done this by combining two well-known, well-studied and standard transforms, namely the Fourier and the Hadamard.

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